ELSEVIER Stochastic Processes and their Applications 54 (1994) 197–213
stochastic processes and their applications

Stochastic models and statistical methods for analysing roughness of plateau-honed surfaces

Peter Hall[a,b,*], David Matthews[a]

[a] *Centre for Mathematics and its Applications, Australian National University, GPO Box 4, Canberra ACT 2601, Australia*
[b] *CSIRO Division of Mathematics and Statistics.*

Received September 1993; revised February 1994

Abstract

Plateau honing is a way of enhancing the lubricant retention qualities of interfacing surfaces which are in relative motion. Surfaces prepared in this way enjoy two levels of roughness, which broadly correspond to a sequence of smooth plateaus at which they touch, and a series of rough crevices honed into one or both of the surfaces for the purpose of retaining lubricant. In the present paper we suggest a stochastic model for a class of such surfaces, based on an alternating sequence of segments of two processes of quite different roughness. Statistical methods are proposed for estimating the principal parameters of these processes, and their performance is studied both theoretically and numerically. We describe roughness in terms of a scale-free quantity, which may be taken equal to either fractal dimension or fractal index, and a measure of scale, the topothesy.

Keywords: Fractal dimension; Fractal index; Gaussian process; Lubrication; Plateau-honed surface; Roughness; Smoothness; Threshold; Topothesy

AMS (1991) SUBJECT CLASSIFICATION: 62M99, Secondary 62G05

1. Introduction

In the study of lubricants and of their effect on friction between two surfaces in relative motion, surface roughness plays a significant role. Broadly speaking, the level of friction between the two surfaces decreases with decreasing roughness. However, the range over which this relationship is valid is restricted by a number of interrelated factors. One of these is temperature, another lubricant type, and a third is the ability of the surfaces to offer places for the lubricant to collect. If the surfaces are extremely smooth then, as their temperature increases, the lubricant will gradually move away from the interface as nothing is available to retain it. In crude terms the lubricant requires a place to congregate, to counteract its tendency to slip out of the interface

*Corresponding author.

SSDI 0304-4149(94)00019-P

owing to influences of temperature, pressure and motion of the surfaces. The lubricant becomes much less viscous as temperature increases, and any low-temperature effects which "stickiness" might confer, allowing the lubricant to be retained at the interface, rapidly disappear.

Thus, an "ideal" lubricated surface should have some element of roughness, to retain the lubricant. The surface should be relatively smooth in the vicinity of its modal height. where the lubricant may collect. Ceramic surfaces achieve this end by having a granular structure, in which the grains are smooth and the crevices between them provide a niche for lubricants. However, a highly polished metal surface does not usually enjoy such a natural opportunity for lubrication. To overcome this problem the polished surface is often slightly roughened, rather like the effect of applying a few light strokes of sandpaper to a lacquered timber surface. This roughening is not applied assiduously, and much of the smooth surface is left untouched. The result is a plateau-honed surface, which retains the character of the original smooth surface at the plateau where it interfaces with the other surface, but has been honed to contain crevices in which the lubricant may collect.

The only plateau-honed surface data of which we are aware were recorded by optical and stylus profilometers along line transects. Therefore, the methods in this paper will be developed for line transect data, and we shall model only line transects of random fields. However, it will be clear that our models and methods have straightforward generalization to bivariate data, should these be available.

For the sake of definiteness and simple exposition we model the abstract surfaces by Gaussian processes. This approach permits fractal dimension to be employed as a scale-free measure of roughness, following an approach which is frequently used in closely related applications (e.g. Berry and Hannay 1978, Berry 1979, Gilbertson and Zipp 1981, Coster and Chermant 1983, Mandelbrot et al., 1984, Carter et al., 1988, Thomas and Thomas 1988, Ogata and Katsura 1991, Taylor and Taylor 1991). In particular, it allows fractal dimension to be represented very simply in terms of fractal index. However, Gaussian processes are by no means alone in permitting this simple relationship between dimension and index; see Hall and Roy (1994). Our approach has quite general application, and indeed closely related techniques may be used to treat smooth functions of Gaussian processes, such as chi-squared processes, that can be employed to model asymmetric (e.g. worn) surfaces.

Our statistical methods for estimating fractal dimension and index are based on calculating differences of surface height on a fine grid, and classifying the differences into two classes according to a threshold rule. If the threshold is chosen appropriately then the respective classes consist primarily of differences which come purely from the rough surface and purely from the smooth surface. Some contamination of these classes, by pure differences of the wrong type or by hybrid differences, is inevitable. This contamination contributes to estimator bias, and can be reduced by refining the basic threshold rule.

Section 2 will introduce our models and describe their main features. Section 3 will introduce our statistical methods based on thresholding. Refinements will be considered in Section 4, which will also present numerical work. Finally, proofs of technical arguments will be given in Section 5.

2. Gaussian-based model for plateau-honed surface

Let X_1, X_2 denote independent stationary Gaussian processes on $(-\infty, \infty)$ with respective covariance functions $\gamma_j(t) = \operatorname{cov}\{X_j(0), X_j(t)\}$. Our model regards the height (at position t) of a line transect across the surface as being representable by $X(t) = \min\{X_1(t), X_2(t) + a\}$, where X_1 and X_2 are two stochastically independent self-similar processes with different levels of roughness, and a is a variable parameter. We may think of X_2 as representing a line transect of the relatively rough surface of the honing tool. It is brought into contact with the smoother surface, of which X_1 denotes a line transect. The constant a represents the amount of pressure placed on the honing tool. The trace denoted by X is comprised of an alternating sequence of segments from X_1 and X_2, in which the proportion of the X_1 process is

$$p = P\{X_1(0) < X_2(0) + a\}.$$

In view of the irregularity of sample paths of fractal processes, if more than one of the processes X_1 and X_2 is represented in any given interval then, with probability one, there is an infinite number of changes from X_1 to X_2, and back again, within that interval.

We assume, without loss of generality, that both X_1 and X_2 processes have zero means. Any difference between the centres of these processes, or in practical terms between the pressures placed on the honing tools which produce the two levels of roughness, is expressed by the constant a.

We shall describe the roughness of X_j by its fractal index α_j, which in turn governs the rate at which $\gamma_j(t)$ converges to $\gamma_j(0)$ as $t \to 0$:

$$\gamma_j(t) = \gamma_j(0) - c_j|t|^{\alpha_j} + o(|t|^{\alpha_j}), \tag{2.1}$$

as $t \to 0$, where $c_j > 0$ and $0 < \alpha_j \leq 2$. The case $\alpha_j = 2$ corresponds to a particularly smooth process X_j, indeed one which has differentiable sample paths. Perhaps surprisingly, this situation is of little practical interest, since even the smoother of the two honing operations generally produces a surface which is far rougher than may reasonably be described by a differentiable function. The fractal index of X is identical to that of the rougher of the two component processes, and indeed it may be shown by direct but tedious calculation that

$$\gamma(t) = \gamma(0) - c_1 p|t|^{\alpha_1} + o(|t|^{\alpha_1}).$$

To first and second orders the scale of the oscillations of X_j is governed by $\gamma_j(0)$ and c_j. For larger values of $\gamma_j(0)$ the sample paths of X_j fluctuate over a wider range; for larger values of c_j the local effects of correlation among fluctuations are weaker, and the fluctuations appear somewhat more erratic. The fractal dimension D_j of sample paths of X_j is expressible very simply in terms of α_j: $D_j = 2 - \frac{1}{2}\alpha_j$. See Adler (1981, Chapter 8.) There is little practical interest in cases where the two honing operations produce surfaces of identical roughness, and so we shall assume without loss of generality that $\alpha_1 < \alpha_2 (< 2)$, i.e. $D_1 > D_2$. Thus, the sample paths of X_1 will be supposed rougher than those of X_2.

Since the roughnesses of X_1 and X_2, as represented by their fractal dimensions, are expressible very simply through α_1 and α_2, then the latter are obvious candidates for estimation. If $\hat{\alpha}_j$ is an estimator of α_j then $\hat{D}_j = 2 - \frac{1}{2}\hat{\alpha}_j$ is an estimator of D_j. Additionally, the quantity c_j, sometimes called the topothesy of X_j, contains important information about the scale of X_j, complementing the scale-free quantity α_j. It too requires estimation.

Of course, the fractal dimension of X is not well defined. At a local level the fractal dimension of a sample path of X oscillates between D_1 and D_2, depending on which of X_1 and X_2 is present at a particular point in the realization. Thus, the classic relationship $D = 2 - \frac{1}{2}\alpha$, between fractal dimension and fractal index, does not hold for the process X.

We shall assume that X is observed at points on a sequence of regular grids within a given interval, which we shall take without loss of generality to be $[0, 1]$. The edge width of the kth grid will be taken to be n_k^{-1} for $1 \leq k \leq m$. In practice the grids are usually nested and are not necessarily chosen so that both the ends of the interval $[0, 1]$ are grid points for all grids. However, for ease of notation we shall take the kth grid to be $\{i/n_k, 0 \leq i \leq n_k\}$; it is a trivial matter to derive analogous results for more general grids. We shall write n for a generic n_k.

Let

$$p^* = \int_0^1 I\{X(t) = X_1(t)\}\,\mathrm{d}t$$

denote the proportion of the trace of X on $[0, 1]$ which is a realization of X_1. Of course, $E(p^*) = p$. We shall show that from a simple trace of X on $[0, 1]$ satisfying $0 < p^* < 1$, we may consistently estimate c_1, c_2, α_1 and α_2 by examining the sample path on increasingly fine grids. Thus, our asymptotic theory involves grid width shrinking to zero while the interval $[0, 1]$ remains fixed. This allows us to make the didactic point that consistent estimation of α_1 and α_2 does not require an infinite trace of X. If we were also to allow the size of the interval to increase then we could consistently estimate p, $\gamma_1(0)$ and $\gamma_2(0)$. In principle the latter two quantities can be estimated from a finite trace, but this fact relies crucially on the assumption that X_1 and X_2 are both Gaussian, which is not quite so critical for inference about the other parameters of interest.

It may seem a little odd that the quantities c_1, c_2, α_1 and α_2 can be estimated consistently from a trace of the process X over a fixed interval, so we pause here to give an intuitive explanation. For the sake of simplicity we focus on the case where we have a homogeneous process X whose covariance function γ satisfies $\gamma(t) = \gamma(0) - c|t|^\alpha + o(|t|^\alpha)$, rather than the more complex process $\max(X_1, X_2 + a)$ discussed above. However, it will be clear that our argument applies more generally. Assume that $\alpha < 2$, or equivalently that $D > 1$. Then, provided only that the fixed numbers t_1 and t_2 are distinct, the correlation of the random variables $X(t_1 + h) - X(t_1)$ and $X(t_2 + h) - X(t_2)$ tends to 0 as $h \to 0$, so they are asymptotically independent. There is an arbitrarily large number of such differences for sufficiently small h. Thus, there is an infinite amount of information in arbitrarily fine

differences about the behaviour of γ near the origin. It is that information which we use to consistently estimate γ in the neighbourhood of the origin, and hence to estimate c and α from a trace of the process on a fixed interval, arbitrarily small.

In practice, several passes over the data would be required to estimate c_1, c_2, α_1 and α_2. This is because the form of the estimators depends to some extent on having an approximate idea of the value α_1 and α_2, which is often best gained by iterating the estimation procedure.

3. Threshold-based estimators of $c_1, c_2, \alpha_1, \alpha_2$

Let $Y_{ni} = X(i/n) - X\{(i-1)/n\}$, $1 \le i \le n$, denote the differences on the grid $\{0, 1/n, 2/n, \ldots, 1\}$. We say that Y_{ni} is pure, of type $j = 1$ or 2, if both $X\{(i-1)/n\} = X_j\{(i-1)/n\}$ and $X(i/n) = X_j(i/n)$. If a difference is not pure then we call it a hybrid difference. Divide the differences into two classes, $\mathscr{C}_1$ and $\mathscr{C}_2$, depending on whether $|Y_{ni}| > \tau$ or $|Y_{ni}| \le \tau$, respectively. Here τ denotes a threshold, and is intended to represent a numerical value which is a reasonable discriminator between differences which are pure of type 1 and those which are pure of type 2. (In this context, hybrid differences are a nuisance. They contribute to the biases of all our estimators, and ideally we would like to identify them and remove them.) We shall take $\tau = n^{-\alpha/2}$ where $0 \le \alpha \le \infty$. This represents an idealized, asymptotic version of thresholds that would be used in practice. Our theory goes through without change if τ is altered to $Cn^{-\alpha/2}$ for an arbitrary but fixed $C > 0$.

The variogram computed from differences in the class $\mathscr{C}_j$, with lag equal to the grid edge-width, is defined by

$$S_j = S_j(n) = n^{-1} \sum_{Y_{ni} \in \mathscr{C}_j} Y_{ni}^2, \quad j = 1, 2. \tag{3.1}$$

For many choices of α it is true that as $n \to \infty$,

$$S_1(n) = D_1 p^* n^{-\beta_1} + o_p(n^{-\beta_1}), \quad S_2(n) = D_2(1 - p^*) n^{-\beta_2} + o_p(n^{-\beta_2}), \tag{3.2}$$

where D_j, $\beta_j > 0$ are constants. If $0 < p^* < 1$ then $\beta_1, \beta_2, C_1 = D_1 p^*$ and $C_2 = D_2(1 - p^*)$ may be estimated from S_1 and S_2 by simple linear regression, as follows. Fix an integer $m \ge 2$ and let $n_1 < \cdots < n_m$ be integers diverging to infinity and such that $n_{k-1}/n_k \to l_k > 0$. Compute the regression line of $\log S_j(n_k)$ on $\log n_k$, $1 \le k \le m$, and let $\log \hat{C}_j$ and $-\hat{\beta}_j$ denote its intercept and slope, respectively. Then $\hat{C}_j \to C_j$ and $\hat{\beta}_j \to \beta_j$ in probability, as the n_k's diverge. This result is immediate from (3.2), provided we assume that both the X_1 and X_2 processes make nontrivial contributions to the observed trace of the process X. If α is chosen appropriately then consistent estimators of $c_1, c_2, \alpha_1, \alpha_2$ may be computed as elementary functions of $\hat{C}_1, \hat{C}_2, \hat{\beta}_1, \hat{\beta}_2$. There are several possibilities, of which the simplest will be noted in Remark 3.4.

The effectiveness of this approach depends very much on choice of α. If $0 \le \alpha < \alpha_1$ then D_1 and β_1 are not well-defined, since with probability tending to 1 the class $\mathscr{C}_1$ is

empty. If $\alpha_1 \leq \alpha \leq \infty$ then $\beta_1 = \alpha_1$, and so α_1 may be estimated consistently by regression of $\log S_1(n)$ on $\log n$. The range of values taken by β_2 is rather more complex:

$$\beta_2 = \begin{cases} \alpha_1 & \text{if } 0 \leq \alpha \leq \alpha_1, \\ \frac{1}{2}(3\alpha - \alpha_1) & \text{if } \alpha_1 < \alpha \leq \frac{1}{3}(\alpha_1 + 2\alpha_2), \\ \alpha_2 & \text{if } \frac{1}{3}(\alpha_1 + 2\alpha_2) < \alpha \leq \alpha_2, \\ \alpha & \text{if } \alpha_2 < \alpha < \alpha_2 + 2. \end{cases} \tag{3.3}$$

The expansion of S_2 in (3.2) does not hold for any D_2 or β_2 if $\alpha \geq \alpha_2 + 2$. Indeed, if $\alpha > \alpha_2 + 2$ then the chance that $\mathscr{C}_2$ is empty converges to 1 as $n \to \infty$, while if $\alpha = \alpha_2 + 2$ then $\mathscr{C}_2$ contains only $O_p(1)$ elements.

It may be seen from these results that consistent estimators of α_1 and α_2 may be obtained by thresholding at a number of different τ's and analysing the results. In a sense the most appropriate range for α is $\frac{1}{3}(\alpha_1 + 2\alpha_2) < \alpha < \alpha_2$. If α lies in this interval and $\tau = n^{-\alpha}$ then the slope of the regression line of $\log S_j(n_k)$ on $\log n_k$ is a consistent estimator of $-\alpha_j$, for both $j = 1, 2$. In effect, this threshold divides the differences into classes $\mathscr{C}_1$ and $\mathscr{C}_2$ such that $\mathscr{C}_1$ consists almost entirely of pure type 1 differences and $\mathscr{C}_2$ of pure type 2 differences. The former are of size $n^{-\alpha_1/2}$ and the latter of size $n^{-\alpha_2/2}$, roughly speaking, and so S_j is of size $n^{-\alpha_j}$, as indicated by (3.2) with $\beta_j = \alpha_j$.

Of course, the threshold method does not produce perfect classification along these ideal lines. Even if α lies in the "ideal" range, $\frac{1}{3}(\alpha_1 + 2\alpha_2) < \alpha < \alpha_2$, the class $\mathscr{C}_2$ is contaminated by pure type 1 differences whose presence causes some deterioration in the performance of $\hat{\alpha}_2 = \hat{\beta}_2$. This contamination is the reason why $\beta_2 < \alpha_2$, rather than equal to α_2, when $\alpha < \frac{1}{3}(\alpha_1 + 2\alpha_2)$—for such α's the Y_{ni}'s in $\mathscr{C}_2$ which are pure of type 1 make such a large contribution that they render the statistic $\hat{\beta}_2$ inconsistent for α_2. The estimator of α_1, based on differences in $\mathscr{C}_1$, fares much better in comparison. Details will be given in remarks following our main theorems.

Next we formalize the claims made earlier about properties of S_1 and S_2. Throughout we assume the Gaussian process model described in Section 2, in particular expansion (2.1) of the covariance functions γ_1 and γ_2. We suppose that $0 < \alpha_1 < \alpha_2 < 2$, and write N for a standard normal random variable. Theorem 3.j describes properties of S_j for $j = 1, 2$.

Theorem 3.1. *If $0 \leq \alpha \leq \alpha_1$ then $P(\mathscr{C}_1$ is empty$) \to 1$ as $n \to \infty$, and so $P(S_1 = 0) \to 1$. If $\alpha \geq \alpha_1$ then*

$$S_1 = D_1 p^* n^{-\alpha_1} + o_p(n^{-\alpha_1}), \tag{3.4}$$

where $D_1 = 2c_1\, E[N^2 I\{|N| > (2c_1)^{-1/2}\}]$ if $\alpha = \alpha_1$ and $D_1 = 2c_1$ if $\alpha > \alpha_1$.

Theorem 3.2. *If $0 \leq \alpha < \alpha_2 + 2$ then*

$$S_2 = C_2 n^{-\beta_2} + o_p(n^{-\beta_2}), \tag{3.5}$$

where β_2 *is given by* (3.3) *and*

$$C_2 = \begin{cases} 2c_1 p^* & \text{if } 0 \le \alpha < \alpha_1, \\ 2c_2\, E[N^2 I\{|N| \le (2c_1)^{-1/2}\}]p^* & \text{if } \alpha = \alpha_1, \\ \frac{1}{3}(\pi c_1)^{-1/2} p^* & \text{if } \alpha_1 < \alpha < \frac{1}{3}(\alpha_1 + 2\alpha_2), \\ \frac{1}{3}(\pi c_1)^{-1/2} p^* + 2c_2(1 - p^*) & \text{if } \alpha = \frac{1}{3}(\alpha_1 + 2\alpha_2), \\ 2c_2(1 - p^*) & \text{if } \frac{1}{3}(\alpha_1 + 2\alpha_2) < \alpha < \alpha_2, \\ 2c_2\, E[N^2 I\{|N| \le (2c_2)^{-1/2}\}](1 - p^*) & \text{if } \alpha = \alpha_2, \\ \frac{1}{3}(2c_2)^{-1/2}(1 - p^*) & \text{if } \alpha_2 < \alpha < \alpha_2 + 2. \end{cases}$$

If $\alpha = \alpha_2 + 2$ *then* $n^{\alpha_2 + 2} S_2$ *has a proper, nondegenerate limiting distribution, and if* $\alpha > \alpha_2 + 2$ *then* $P(S_2 = 0) \to 1$.

Theorem 3.3. *If* $\alpha_1 < \alpha < \alpha_2$ *then* $\hat{p}^* \equiv n^{-1}(\#\mathscr{C}_1) = p^* + o_p(1)$ *as* $n \to \infty$.

Remark 3.1. Errors in classifying differences into the classes $\mathscr{C}_1$ and $\mathscr{C}_2$ can significantly affect performance of the estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$. Provided that $\alpha_1 < \alpha < \alpha_2$, the probability that $\mathscr{C}_1$ contains no pure type 2 differences converges to 1 as $n \to \infty$. However, if $\alpha \ge \alpha_2$ then the contribution to S_1 from pure type 2 differences is of size $n^{-\alpha_2}$, which results in a term of size $n^{-(\alpha_2 - \alpha_1)}$ in the bias of the estimator $\hat{\alpha}_1 = \hat{\beta}_1$. This can be important, depending on the relative values of α_1 and α_2. If $\alpha > \alpha_1$ then the presence of hybrid differences in $\mathscr{C}_1$ results in a contribution of size $n^{-3\alpha_1/2}$ to S_1, producing a term of size $n^{-\alpha_1/2}$ in the bias of $\hat{\alpha}_1$. This can be important for small α_1.

Remark 3.2. The contributions to S_2 from pure type 1 differences and pure type 2 differences are of sizes $n^{-(3\alpha - \alpha_1)/2}$ and $n^{-\alpha_2}$, respectively. The latter dominates if and only if $\alpha > \frac{1}{3}(\alpha_1 + 2\alpha_2)$, but even in this case the bias of the estimator $\hat{\alpha}_2 = \hat{\beta}_2$ includes a term of size $n^{-(3\alpha - \alpha_1 - 2\alpha_2)/2}$, resulting from the misclassified pure type 1 differences. This quantity can be significant if α is not sufficiently greater than $\frac{1}{3}(\alpha_1 + 2\alpha_2)$. If $\alpha_1 < \alpha < \alpha_2$ then the contribution of hybrid differences to S_2 is of size $n^{-3\alpha/2}$, which is negligible relative to the contribution of pure type 1 differences.

Remark 3.3. In practice the estimation of α_1 and α_2 using these methods would require several passes over the data, using a range of different thresholds as well as different grid edge widths. Such an approach would allow one to gradually acquire information about the ranges in which α_1 and α_2 appear to lie, before attempting a more definitive analysis. There is of course no reason why the same threshold need be used to estimate both α_1 and α_2.

Remark 3.4. Combining the results of Theorems 3.1–3.3 we see that if $0 < p^* < 1$ and $\frac{1}{3}(\alpha_1 + 2\alpha_2) < \alpha < \alpha_2$ then the regression-based estimators $\hat{C}_1, \hat{C}_2, \hat{\beta}_1, \hat{\beta}_2$ have the property that $\hat{c}_1 \equiv \hat{C}_1/2\hat{p}^* \to c_1$, $\hat{c}_2 \equiv \hat{C}_2/2(1 - \hat{p}^*) \to c_2$, $\hat{\alpha}_1 \equiv \hat{\beta}_1 \to \alpha_1$ and $\hat{\alpha}_2 \equiv \hat{\beta}_2 \to \alpha_2$ in probability. Of course, these results represent only examples of more general limit theorems of the same type, for different values of α.

Remark 3.5. The "in probability" qualification of all the above convergence statements may be strengthened to "with probability one". We have not done so only because we wish to minimize the lengths of our proofs.

Remark 3.6. Variogram methods for estimating the fractal dimension of a "pure" process, in which any part of a sample path has with probability one the same fractal dimension regardless of location, have been discussed by Constantine and Hall (1994). Of course, no threshold arguments are required in that context.

4. Simulation study

The theoretical results discussed previously were confirmed in a simulation study, which took the following form. Realizations of the process $X = \min(X_1, X_2 + a)$ were generated on a grid. Here, X_1 and X_2 were taken to be independent, stationary Gaussian processes with zero means and covariance functions γ_1 and γ_2, respectively, where $\gamma_j(t) = \exp(-c_j\|t\|^{\alpha_j})$, $c_j > 0$ and $0 < \alpha_j < 2$. The processes were generated using an efficient method proposed by Wood and Chan (1993). In each simulation the constant a was chosen so that the long-run average proportion of the trace that was X_1 rather than $X_2 + a$ was $\frac{1}{2}$.

Clearly, the performance of $\hat{\alpha}_1$ and $\hat{\alpha}_2$ had to be considered for different values of α_1, α_2, c_1, c_2 and n_k. All combinations of $\alpha_1 = 0.1, 0.2, 0.3, 0.4, 0.5, 0.75, 1.0, 1.25$; $\alpha_2 = 1.3, 1.5, 1.6, 1.75, 1.8, 1.85, 1.9, 1.95$; $c_1 = 0.5, 1, 2, 4, 8, 16$; $c_2 = 0.5, 1, 2, 4, 8, 16$; and $n_k = 2000, 2200, 2400, 2600, 2800, 3000$ were considered. For each set of combinations, values of the threshold τ were selected from the interval $[0, 2]$.

For each set of combinations, $X(t)$ was generated on the grid $(0, n_k^{-1}, \ldots, 1)$ and the variograms $S_1(n_k)$, $S_2(n_k)$ were formed. The estimators $\hat{\alpha}_1$ and $\hat{\alpha}_2$ were then calculated N times, each time computing an estimate $\hat{\alpha}_{ji}$ of α_j. The mean squared error of $\hat{\alpha}_j$ was approximated numerically by

$$\widetilde{MSE}(\hat{\alpha}_j) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha}_{ji} - \alpha_j)^2. \tag{4.1}$$

In the study N was taken to be 30.

These mean squared errors were calculated for the different values of τ. The results are shown in Figs. 1 and 2, where they are compared with the mean squared errors of a more sophisticated method which we shall discuss shortly.

The allocations to $\mathscr{C}_1$ and $\mathscr{C}_2$ resulting from using the original method were examined in each simulation. The allocation of pure type 2 differences to $\mathscr{C}_1$ was virtually zero until τ became small. As τ further decreased in size the contributions of pure type 2 differences became very large – almost all of the differences were allocated to $\mathscr{C}_1$. For hybrid differences, as τ decreased the contributions to $\mathscr{C}_1$ became larger.

In the case of $\mathscr{C}_2$, as the size of τ decreased the number of allocations to $\mathscr{C}_2$ also dropped, with the size of $\mathscr{C}_2$ eventually decreasing to 0 for very small τ. For large

values of τ, $\mathscr{C}_2$ had large proportions of pure type 1 and hybrid differences allocated to it. But as τ decreased in size the contamination of $\mathscr{C}_2$ decreased until $\mathscr{C}_2$ contained only pure type 2 differences. The size of $\mathscr{C}_2$ then decreased as τ decreased until eventually $\mathscr{C}_2$ was empty.

We developed the allocation method further, with the aim of increasing the proportion of pure type 1 differences allocated to $\mathscr{C}_1$ and pure type 2 differences allocated to $\mathscr{C}_2$. Our goal was to discard as many hybrid differences as possible, leaving mainly pure differences to be allocated. In practice it is quite difficult to identify hybrid differences.

The strategy we adopted was to reduce bias (or systematic error) by removing hybrid differences, at the expense of increasing standard deviation (or stochastic error) by deleting some of the pure differences as well. We used Monte Carlo simulation to assess the balance of reduced bias versus increased variance.

After much experimentation we settled on a method which involved examining "runs" of differences of the same type. These were classified as $\mathscr{C}_1$ or $\mathscr{C}_2$, the idea being that in a run of differences all allocated to $\mathscr{C}_1$ or $\mathscr{C}_2$ the probability that any of these differences is hybrid is less than that for a sequence of differences which were not identically allocated.

This method allocates a difference Y_{ni} to $\mathscr{C}_1$ or $\mathscr{C}_2$ only if that difference occurs in a run of a certain length. Any difference which does not occur in this fashion is discarded, using the following rules.

1. For a given value of τ, form the differences Y_{ni}.
2. Allocate a group of differences $Y_{ni}, Y_{n,i+1}, \ldots, Y_{n,i+(L-1)}$ to $\mathscr{C}_1$ if $|Y_{n,i+j}| > \tau$, for all $j = 0, \ldots, L-1$. Allocate the differences to $\mathscr{C}_2$ if each $|Y_{n,i+j}| \leq \tau$.
3. Discard all differences which do not occur in a run of length L.
4. Redefine the variograms S_1 and S_2 in the obvious way, based on the numbers of pure differences allocated by this new rule.

This new method increases the *proportions* of correctly classified differences, but reduces the *numbers* of those differences. In that sense, information is lost. Indeed, mean squared error tends to decrease in cases where the estimator previously suffered from severe bias problems, but tends to increase elsewhere. See Figs. 1 and 2, but note that because a large number of different parameters were simulated, only a subset of them and their mean square errors have been displayed.

Fig. 1 shows how $\hat{\alpha}_1$ performed when calculated from $\mathscr{C}_1$ and Fig. 2 shows how $\hat{\alpha}_2$ performed. The horizontal axis of both figures is plotted on a logarithmic scale, since this seemed the best way of including very small and very large values of τ on the one graph. The dotted line and solid line indicate mean squared errors under the old and new methods, respectively.

In all cases the new method and original method converge quickly, as τ decreases, to having the same mean squared error, even though for large values of τ there was often quite a difference, with the original method having lower mean squared error. Consistently the optimal choice of τ was very small. For estimation of α_1 from the differences in $\mathscr{C}_1$ the new method does not appear to be significantly better than the original.

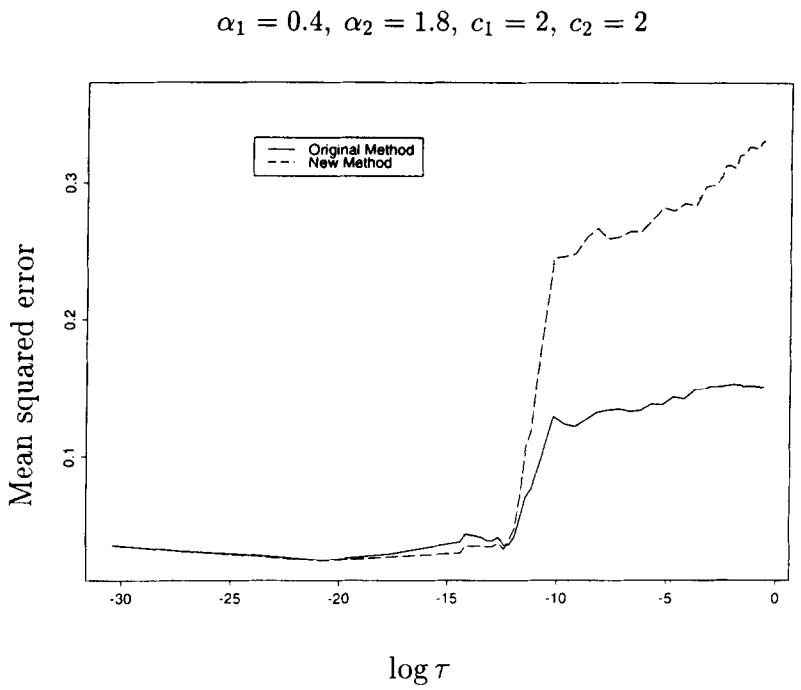

Fig. 1. Plots of mean square errors for α_1 estimated from C_1. The class C_1 was empty for large τ and thus no estimates of α_1 could be obtained. For smaller values of τ the original method gave much better results than the new method. For very small values of τ the mean squared errors of both methods were very close together. Very small values of τ gave the smallest mean squared errors.

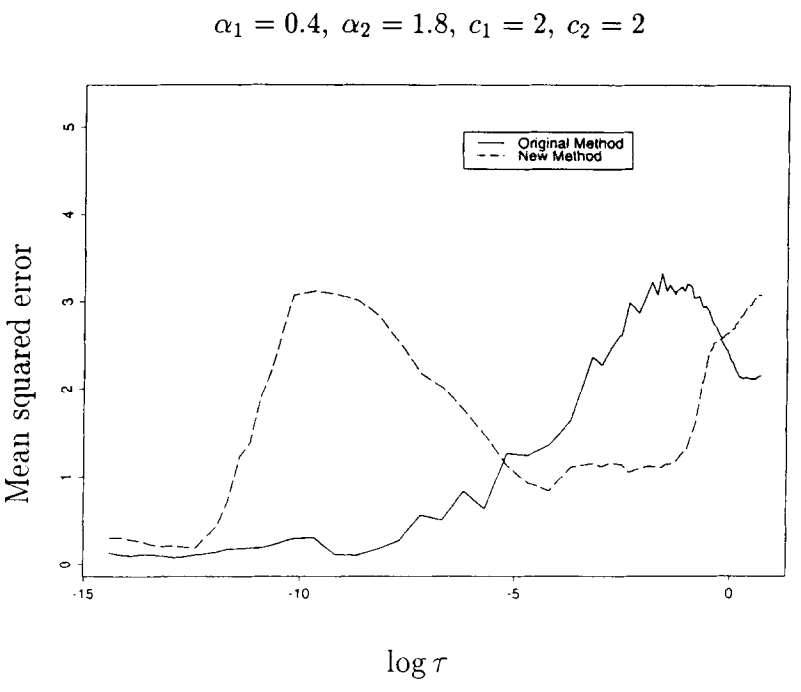

Fig. 2. Plots of mean square errors for α_2 estimated from C_2. The original method produced the best results when small values of τ were used. For increased values of τ both methods gave large errors.

The new method offers the greatest improvement on the original method when $\hat{\alpha}_1$ is computed from the differences in $\mathscr{C}_2$. For large values of τ the new method consistently gives reasonable estimates of α_1, whereas the original method performs poorly. The original method does give good estimates for some very small values of τ, but these estimates were not generally better than those obtained using the new method and large values of τ. However, the performance of $\hat{\alpha}_1$ is generally superior when the estimator is computed from $\mathscr{C}_1$ rather than $\mathscr{C}_2$.

The estimator $\hat{\alpha}_2$ performs best when the original method is used. The best results using the estimator $\hat{\alpha}_1$ were obtained with differences from the class $\mathscr{C}_1$. The estimator of α_1 is generally more accurate than that of α_2.

Overall the best results were obtained from the original method.

5. Proofs

Proof of Theorem 3.1. Let N denote a standard normal random variable and B a generic positive constant. Put $u = n^{-1}$.

(a) *If* $0 \le \alpha < \alpha_1$ *then* $P(\mathscr{C}_1 \text{ is empty}) \to 1$. Observe that

$$P(\mathscr{C}_1 \text{ is nonempty}) \le n \sum_{j=1}^{2} P\{|X_j(u) - X_j(0)| n^{-\alpha/2}\} + 2nq_n$$

$$\le n \sum_{j=1}^{2} P(|N| > Bn^{(\alpha_j - \alpha)/2}) + 2nq_n, \tag{5.1}$$

where

$$q_n = P\{X_1(0) < X_2(0) + a, X_1(u) > X_2(u) + a, |X_1(0) - X_2(u) - a| > n^{-\alpha/2}\}.$$

(The two-term series on the right-hand side of (5.1) accounts for pure differences, and $2nq_n$ allows for hybrid differences.) Put $Z_0 = X_1(0) - X_2(0) - a$, $Z_j = X_j(u) - X_j(0)$. In this notation,

$$\begin{aligned} q_n &= P(Z_0 < 0, Z_0 + Z_1 - Z_2 > 0, |Z_0 - Z_2| > n^{-\alpha/2}) \\ &\le P(|Z_2| > \tfrac{1}{2} n^{-\alpha/2}) + P(Z_0 < 0, Z_0 + Z_1 - Z_2 0, |Z| > \tfrac{1}{2} n^{-\alpha/2}) \\ &\le P(|Z_2| > \tfrac{1}{2} n^{-\alpha/2}) + P(|Z_1 - Z_2| > \tfrac{1}{2} n^{-\alpha/2}) \\ &\le P(|Z_1| > \tfrac{1}{4} n^{-\alpha/2}) + 2P(|Z_2| > \tfrac{1}{4} n^{-\alpha/2}) \\ &\le P(|N| > Bn^{(\alpha_1 - \alpha)/2}) + 2P(|N| > \tfrac{1}{4} n^{(\alpha_2 - \alpha)/2}). \end{aligned} \tag{5.2}$$

Combining (5.1) and (5.2), and noting that $0 \le \alpha < \alpha_1 < \alpha_2$, we conclude that for $B > 0$ sufficiently small,

$$P(\mathscr{C}_1 \text{ is nonempty}) \le 8nP(|N| > Bn^{(\alpha_1 - \alpha)/2}) = o(n^{-\lambda})$$

for all $\lambda > 0$.

(b) *if* $\alpha \ge \alpha_1$ *then* (3.4) *holds with* $C_1 = 2c_1 p^*$ *if* $\alpha > \alpha_1$ *and with* $C_1 = 2c_1 p^* E[N^2 I\{|N| > (2c_1)^{-1/2}\}]$ *if* $\alpha = \alpha_1$. Let $\mathscr{C}_{11}, \mathscr{C}_{12}, \mathscr{C}_{13}$ denote, respectively, the set of

pure type 1 differences in $\mathscr{C}_1$, pure type 2 differences in $\mathscr{C}_1$, and hybrid differences in $\mathscr{C}_1$. Then $\mathscr{C}_{11}, \mathscr{C}_{12}, \mathscr{C}_{13}$ are disjoint, with union $\mathscr{C}_1$. Our derivation of Part (b) from this point involves four steps.

Step (i). *For* $\alpha \geq \alpha_1$,

$$n^{-1} \sum_{Y_{ni} \in \mathscr{C}_{11}} Y_{ni}^2 = C_1 n^{-\alpha_1} + o_p(n^{-\alpha_1}).$$

Put $Z(t) = X_1(t) - X_2(t) - a$, let Z_i be as before, and define $D_1 = 2c_1$ if $\alpha > \alpha_1$, $D_1 = 2c_1\, E[N^2 I\{|N| > (2c_1)^{-1/2}\}]$ if $\alpha = \alpha_1$. In this notation,

$$\begin{aligned}
\mu &\equiv E\{Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{11})\} = E[Z_1^2 I\{Z(0) < 0, Z(u) < 0, |Z_1| > n^{-\alpha/2}\}] \\
&\sim E[Z_1^2 I(|Z_1| > n^{-\alpha/2})]\, P\{Z(0) < 0\} \sim D_1 p n^{-\alpha_1}, \\
\mu_{i_1 i_2} &\equiv E\{Y_{ni_1}^2 I(Y_{ni_1} \in \mathscr{C}_{11}) \cdot Y_{ni_2}^2 I(Y_{ni_2} \in \mathscr{C}_{11})\} \leq E(Z_1^4) = O(n^{-2\alpha_1}). \\
\nu_{i_1 i_2} &\equiv E[Y_{ni_1}^2 I(Y_{ni_1} \in \mathscr{C}_{11}) \cdot I\{Z(i_2/n) < 0\}] \leq E(Z_1^2) = O(n^{-\alpha_1}). \\
p_{i_1 i_2} &\equiv P\{Z(i_1/n) < 0, Z(i_2/n) < 0\} \leq 1.
\end{aligned}$$

and with $\sup_{(\varepsilon)}$ denoting the supremum over $1 \leq i_1, i_2 \leq n$ satisfying $|i_1 - i_2| > \varepsilon n$ for $\varepsilon \in (0, 1)$,

$$\sup_{(\varepsilon)} |\mu_{i_1 i_2} - \mu^2| = o(n^{-2\alpha_1}), \quad \sup_{(\varepsilon)} |\nu_{i_1 i_2} - \mu p| = o(n^{-\alpha_1}),$$

$$\sup_{(\varepsilon)} |p_{i_1 i_2} - p^2| = o(1). \tag{5.3}$$

It follows that

$$\begin{aligned}
&E\left(\sum_{i=1}^{n} [Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{11}) - \mu p^{-1} I\{Z(i/n) < 0\}] \right)^2 \\
&= \sum_{i_1} \sum_{i_2} (\mu_{i_1 i_2} + \mu^2 p^{-2} p_{i_1 i_2} - 2\mu p^{-1} \nu_{i_1 i_2}) = o(n^{2-2\alpha_1}),
\end{aligned} \tag{5.4}$$

whence, since $n^{-1} \sum_i I\{Z(i/n) < 0\} \to 1$ in probability,

$$\begin{aligned}
n^{-1} \sum_{i=1}^{n} Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{11}) &= \mu p^{-1} n^{-1} \sum_{i=1}^{n} I\{Z(i/n) < 0\} + o_p(n^{-\alpha_1}) \\
&= \mu p^{-1} p^* + o_p(n^{-\alpha_1}) \\
&= D_1 p^* n^{-\alpha_1} + o_p(n^{-\alpha_1}),
\end{aligned} \tag{5.5}$$

which is the required result.

Step (ii). *For all* $\alpha \geq 0$, *the contribution of* $\mathscr{C}_{12}$ *to* S_1 *equals* $O_p(n^{-\alpha_2})$. If Y_{ni} is pure of type 2 then $Y_{ni} = X_2(i/n) - X_2\{(i-1)/n\}$. Therefore,

$$E\left(n^{-1} \sum_{Y_{ni} \in \mathscr{C}_{12}} Y_{ni}^2 \right) \leq E\{X_2(u) - X_2(0)\}^2 = O(n^{-\alpha_2}).$$

(If $\alpha < \alpha_2$ then the probability that $\mathscr{C}_{12}$ is nonempty converges to zero at rate $n^{-\lambda}$ for each $\lambda > 0$. This verifies one of the points made in Remark 3.1.)

Step (iii). *For all* $\alpha > \alpha_1$, *the contribution of* $\mathscr{C}_{13}$ *to* S_1 *equals* $O_p(n^{-3\alpha_1/2})$. We shall actually prove more – that the expected value, ξ, of the total contribution to $S_1 + S_2$ from all hybrid differences equals $O(n^{-3\alpha_1/2})$. Observe that

$$\xi = 2E[\{X_1(0) - X_2(u) - a\}^2 I\{X_1(0) < X_2(0) + a, X_1(u) > X_2(u) + a\}].$$

If $X_1(0) < X_2(0) + a$ and $X_1(u) > X_2(u) + a$ then

$$X_1(0) - X_1(u) < X_1(0) - X_2(u) - a < X_2(0) - X_2(u),$$

and so

$$\xi \le 2 \sum_{j=1}^{2} E[\{X_j(u) - X_j(0)\}^2 I\{X_1(0) < X_2(0) + a, X_1(u) > X_2(u) + a\}].$$

Let Z_0 and Z_1 be as before. Then for all $\varepsilon, \lambda > 0$,

$$\begin{aligned}
&E[\{X_1(u) - X_1(0)\}^2 I\{X_1(0) < X_2(0) + a, X_1(u) > X_2(u) + a\}] \\
&\le E[Z_1^2 I\{Z_0 < 0, X_1(u) > X_2(0) + a - n^{\varepsilon - (\alpha_2/2)}\}] \\
&\quad + E(Z_1^2) P\{|X_2(u) - X_2(0)| > n^{\varepsilon - (\alpha_2/2)}\} \\
&= E\{Z_1^2 I(-Z_1 - n^{\varepsilon - (\alpha_2/2)} < Z_0 < 0)\} + O(n^{-\lambda}) \\
&\le E\{Z_1^2 P(|Z_0| < |Z_1 + n^{\varepsilon - (\alpha_2/2)}| \mid Z_1)\} + O(n^{-\lambda}) \\
&= O\{E(Z_1^2 | Z_1 + n^{\varepsilon - (\alpha_2/2)}|) + n^{-\lambda}\} = O(n^{-3\alpha_1/2}),
\end{aligned}$$

provided that $\varepsilon \le (\alpha_2 - \alpha_1)/2$ and $\lambda > 3\alpha_1/2$. A similar argument may be employed to show that

$$E[\{X_2(u) - X_2(0)\}^2 I\{X_1(0) < X_2(0) + a, X_1(u) > X_2(u) + a\}] = o(n^{-3\alpha_1/2}).$$

Combining these results we see that $\xi = O(n^{-3\alpha_1/2})$, as had to be shown.

Step (iv). *Completion.* Combining the conclusions of Steps (i)–(iii) we see that

$$\begin{aligned}
S_1 &= n^{-1} \sum_{Y_{ni} \in \mathscr{C}_{11}} Y_{ni}^2 + n^{-1} \sum_{Y_{ni} \in \mathscr{C}_{12}} Y_{ni}^2 + n^{-1} \sum_{Y_{ni} \in \mathscr{C}_{13}} Y_{ni}^2 \\
&= \{C_1 n^{-\alpha_1} + o_p(n^{-\alpha_1})\} + O_p(n^{-\alpha_2}) + O_p(n^{-3\alpha_1/2}) \\
&= C_1 n^{-\alpha_1} + o_p(n^{-\alpha_1}),
\end{aligned}$$

as had to be shown.

Proof of Theorem 3.2. Let $N, u, Z(t), Z_1, Z_2$ have the definitions accorded them in the previous proof. Let $\mathscr{C}_{21}, \mathscr{C}_{22}, \mathscr{C}_{23}$ denote, respectively, the set of pure type 1 differences in $\mathscr{C}_2$, pure type 2 differences in $\mathscr{C}_2$, and hybrid differences in $\mathscr{C}_2$. From this point our proof consists of four steps.

Step (i). *For* $0 \le \alpha < \alpha_1 + 2$,

$$n^{-1} \sum_{Y_{ni} \in \mathscr{C}_{21}} Y_{ni}^2 = D_1' p^* n^{-\beta_1'} + o_p(n^{-\beta_1'}), \tag{5.6}$$

where

$$D_1' = \begin{cases} 2c_1 & \text{if } 0 \le \alpha < \alpha_1, \\ 2c_1 E[N^2 I\{|N| \le (2c_1)^{-1/2}\}] & \text{if } \alpha = \alpha_1, \\ \frac{1}{3}(\pi c_1)^{-1/2} & \text{if } \alpha_1 < \alpha < \alpha_1 + 2 \end{cases}$$

and

$$\beta_1' = \begin{cases} \alpha_1 & \text{if } 0 \le \alpha < \alpha_1, \\ \frac{1}{2}(3\alpha - \alpha_1) & \text{if } \alpha_1 \le \alpha < \alpha_1 + 2; \end{cases}$$

for $\alpha = \alpha_1 + 2$, $n^{\alpha_1 + 2} \sum_{Y_{ni} \in \mathscr{C}_{21}} Y_{ni}^2$ *has a proper, nondegenerate limiting distribution; and for* $\alpha > \alpha_2 + 2$, $P(\mathscr{C}_{21}$ is empty$) \to 1$.

First we prove (5.6) in the case $0 \le \alpha \le \alpha_1$. There,

$$\begin{aligned} \mu &\equiv E\{Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{21})\} = E[Z_1^2 I\{Z(0) < 0, Z(u) < 0, |Z_1| \le n^{-\alpha/2}\}] \\ &\sim E[Z_1^2 I(|Z_1| \le n^{-\alpha/2})]\, P\{Z(0) < 0\} \sim D_1' p n^{-\alpha_1}, \\ \mu_{i_1 i_2} &\equiv E\{Y_{ni_1}^2 I(Y_{ni_1} \in \mathscr{C}_{21}) \cdot Y_{ni_2}^2 I(Y_{ni_2} \in \mathscr{C}_{21})\} \le E(Z_1^4) = O(n^{-2\alpha_1}), \\ \nu_{i_1 i_2} &\equiv E[Y_{ni_1}^2 I(Y_{ni_1} \in \mathscr{C}_{21}) \cdot I\{Z(i_2/n) < 0\}] \le E(Z_1^2) = O(n^{-\alpha_1}), \\ p_{i_1 i_2} &\equiv P\{Z(i_1/n) < 0, \quad Z(i_2/n) < 0\} \le 1, \end{aligned}$$

and for any $0 < \varepsilon < 1$, result (5.3) holds for the new definitions of $\mu_{i_1 i_2}$, $\nu_{i_1 i_2}$, $p_{i_1 i_2}$. It follows as in the proof of Theorem 3.1 that the analogue of (5.4) holds:

$$E\left(\sum_{i=1}^{n} [Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{21}) - \mu p^{-1} I\{Z(i/n) < 0\}] \right)^2 = o(n^{2-2\alpha_1}),$$

whence, in analogy to (5.5), (5.6) is true (for $0 \le \alpha \le \alpha_1$).

To verify (5.6) in the case $\alpha_1 < \alpha < \alpha_1 + 2$, define $V_i = X_1(i/n) - X_1\{(i-1)/n\}$,

$$V_{ni} = n^{\alpha} V_i^2 I(|V_i| \le n^{-\alpha/2}), \quad W_{ni} = V_{ni} I\{Z(i/n) < 0\}.$$

Then $Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{21}) \le n^{-\alpha} W_{ni}$, and

$$\begin{aligned} &E\{n^{-\alpha} W_{ni} - Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{21})\} \\ &= E[Z_1^2 I(|Z_1| \le n^{-\alpha/2}) P\{Z(0) > 0, Z(u) < 0 \mid Z_1\}] \\ &= o[E\{Z_1^2 I(|Z_1| \le n^{-\alpha/2})\}] = o(n^{-(3\alpha - \alpha_1)/2}). \end{aligned}$$

Therefore,

$$n^{-1} \sum_{i=1}^{n} Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{21}) = n^{-1-\alpha} \sum_{i=1}^{n} V_{ni} I\{Z(i/n) < 0\} + o_p(n^{-(3\alpha - \alpha_1)/2}). \tag{5.7}$$

If we show that for any $0 < a_1 < a_2 < 1$,

$$\sum_{a_1 n < i < a_2 n} V_{ni} = D_1'(a_2 - a_1)n^{1-(\alpha-\alpha_1)/2} + o_p(n^{1-(\alpha-\alpha_1)/2}), \tag{5.8}$$

then it will follow from (5.7) that

$$n^{-1}\sum_{i=1}^{n} Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{21}) = D_1' p^* n^{-(3\alpha-\alpha_1)/2} + o_p(n^{-(3\alpha-\alpha_1)/2}),$$

which is (5.6) in the case $\alpha_1 < \alpha < \alpha_1 + 2$. To derive (5.8), first put

$$M = \sum_{a_1 n < i < a_2 n} I(|V_i| \le n^{-\alpha/2})$$

and let $\tilde{\imath}_1 < \tilde{\imath}_2 < \cdots < \tilde{\imath}_m$ denote the successive integers i such that $a_1 n < i < a_2 n$ and $|V_i| \le n^{-\alpha/2}$. Then, provided that $\alpha_1 < \alpha < \alpha_1 + 2$,

$$M = (a_2 - a_1)(\pi c_1)^{-1/2} n^{1-(\alpha-\alpha_1)/2} + o_p(n^{1-(\alpha-\alpha_1)/2}).$$

Write $\langle b \rangle$ for the integer part of $bn^{1-(\alpha-\alpha_1)/2}$, and put $b_0 = (a_2 - a_1)(\pi c_1)^{-1/2}$. Since $|V_{ni}| \le 1$ then for $0 < b < b_0$,

$$\begin{aligned}\left| \sum_{a_1 n < i < a_2 n} V_{ni} - \sum_{j=1}^{\langle b \rangle} V_{n\tilde{\imath}_j} \right| &= \left| \sum_{j=\langle b \rangle + 1}^{M} V_{n\tilde{\imath}_j} \right| \\ &\le (b_0 - b)n^{1-(\alpha-\alpha_1)/2} + o_p(n^{1-(\alpha-\alpha_1)/2}).\end{aligned}$$

Hence, (5.8) will follow if we prove that

$$\sum_{j=1}^{\langle b \rangle} V_{n\tilde{\imath}_j} = \tfrac{1}{3}\langle b \rangle + o_p(n^{1-(\alpha-\alpha_1)/2}).$$

This result may be proved using an argument which is based on the fact that the $V_{n\tilde{\imath}_j}$'s are asymptotically independent and distributed as the square of a variable uniform on (0, 1).

We turn next to the case $\alpha = \alpha_1 + 2$, where

$$P(|Y_{ni}| \le n^{-\alpha/2}) = P(Y_{ni} \le n^{-\alpha/2}) \sim (\pi c_1)^{-1/2} n^{-1}.$$

Hence, noting the asymptotic independence of Y_{ni_1} and Y_{ni_2} for $|i_1 - i_2| > \varepsilon n$, we see that

$$n^{\alpha}\sum_{i=1}^{n} Y_{ni}^2 (Y_{ni} \in \mathscr{C}_{21}) \xrightarrow{\text{dist}} \sum_{i=1}^{L} U_i I\{Z(U_i') < 0\},$$

where L is Poisson-distributed with mean $(\pi c_1)^{-1/2}$, the U_i's and U_i''s are all uniformly distributed in (0, 1), and X_1, X_2, L, the U_i's and the U_i''s are all independent.

Finally, when $\alpha > \alpha_1 + 2$ we have

$$P(\mathscr{C}_{21} \text{ is nonempty}) \le nP(|Z_1| \le n^{-\alpha/2}) = O(n^{1-(\alpha-\alpha_1)/2}) = O(1),$$

as had to be shown.

Step (ii). *For* $0 \leq \alpha < \alpha_2 + 2$,

$$n^{-1} \sum_{Y_{ni} \in \mathscr{C}_{22}} Y_{ni}^2 = D_2'(1 - p^*)n^{-\beta_2'} + o_p(n^{-\beta_2'})$$

where

$$D_2' = \begin{cases} 2c_2 & \text{if } 0 \leq \alpha < \alpha_2, \\ 2c_2 E[N^2 I\{|N| \leq (2c_2)^{-1/2}\}] & \text{if } \alpha = \alpha_2, \\ \frac{1}{3}(\pi c_2)^{-1/2} & \text{if } \alpha_2 < \alpha < \alpha_2 + 2 \end{cases}$$

and

$$\beta_2' = \begin{cases} \alpha_2 & \text{if } 0 \leq \alpha < \alpha_2, \\ \frac{1}{2}(3\alpha - \alpha_2) & \text{if } \alpha_2 \leq \alpha < \alpha_2 + 2; \end{cases}$$

for $\alpha = \alpha_2 + 2$, $n^{\alpha_2 + 2} \sum_{Y_{ni} \in \mathscr{C}_{22}} Y_{ni}^2$ *has a proper, nondegenerate limiting distribution; and for* $\alpha > \alpha_2 + 2$, $P(\mathscr{C}_{22}$ is empty$) \to 1$.

A derivation of these results is similar to that in the case of Step (i), and so will not be given here.

Step (iii). *The contribution of* $\mathscr{C}_{23}$ *to* S_2 *equals* $O_p(n^{-3\alpha_1/2})$ *for* $0 \leq \alpha \leq \alpha_1$, *and equals* $O_p(n^{-3\alpha/2})$ *if* $\alpha \geq \alpha_1$. These results are immediate from the following argument:

$$\begin{aligned} &\tfrac{1}{2}E\{Y_{ni}^2 I(Y_{ni} \in \mathscr{C}_{23})\} \\ &\quad = E[\{X_1(0) - X_2(u) - a\}^2 I\{X_1(0) < X_2(0) + a, \\ &\qquad X_1(u) > X_2(u) + a, |X_1(0) - X_2(u) - a| \leq n^{-\alpha/2}\}] \\ &\quad = E\{(Z_2 - Z)^2 I(Z_2 < Z_2 - Z < Z_1, |Z_2 - Z| \leq n^{-\alpha/2})\} \\ &\quad \leq E(\min\{n^{-\alpha}, \max(Z_1^2, Z_2^2)\} P[|Z_2 - Z| \leq \min\{n^{-\alpha/2}, \max(|Z_1|, |Z_2|)\} | Z_1, Z_2]) \\ &\quad = O(E[\min\{n^{-3\alpha/2}, \max(|Z_1|^3, |Z_2|^3)\}]) \\ &\quad = O\{\min(n^{-3\alpha/2}, n^{-3\alpha_1/2})\}. \end{aligned}$$

Step (iv) (*Conclusion*). Assume initially that $0 \leq \alpha < \alpha_2 + 2$. In this notation we see, on combining the results in Steps (i)–(iii), that

$$\begin{aligned} S_2 &= \left\{ \begin{array}{ll} D_1' p^* n^{-\beta_1'} + o_p(n^{-\beta_1'}) & \text{if } \alpha < \alpha_1 + 2 \\ O_p(n^{-\alpha_1 - 3}) & \text{if } \alpha = \alpha_1 + 2 \\ O & \text{if } \alpha > \alpha_1 + 2 \end{array} \right\} + D_2'(1 - p^*)n^{-\beta_2'} + o_p(n^{-\beta_2'}) \\ &\quad + O_p\{\min(n^{-3\alpha_1/2}, n^{-3\alpha/2})\} \\ &= \begin{cases} D_1' p^* n^{-\beta_1'} + o_p(n^{-\beta_1'}) & \text{if } \beta_1' < \beta_2', \\ \{D_1' p^* + D_2'(1 - p^*)\} n^{-\beta_1'} + o_p(n^{-\beta_1'}) & \text{if } \beta_1' = \beta_2', \\ D_2'(1 - p^*)n^{-\beta_2'} + o_p(n^{-\beta_2'}) & \text{if } \beta_1' > \beta_2', \end{cases} \end{aligned}$$

which implies (3.5). The assertions of Theorem 3.2 for $\alpha \geq \alpha_2 + 1$ follow in a similar way.

Proof of Theorem 3.3. Observe that if $\alpha_1 < \alpha < \alpha_2$ then

$$P(Y_{ni} \text{ is pure of type 1, but in } \mathscr{C}_2) \leq P(|Z_1| \leq n^{-\alpha/2}) = O(n^{-(\alpha-\alpha_1)/2}),$$

$$P(Y_{ni} \text{ is pure of type 2, but in } \mathscr{C}_1) \leq P(|Z_2| > n^{-\alpha/2}) = O(n^{-\lambda}),$$

for all $\lambda > 0$, and

$$P(Y_{ni} \text{ is hybrid}) = 2P(Z < 0, Z + Z_1 - Z_2 > 0) = 2P(Z_2 - Z_1 < Z < 0)$$
$$= O(n^{-\alpha_1/2}).$$

Therefore, if K denotes the total number of differences Y_{ni}, $1 \leq i \leq n$, which are either hybrid, or pure but misclassified, then

$$E(K) = O\{n(n^{-(\alpha-\alpha_1)/2} + n^{-\lambda} + n^{-\alpha_1/2})\} = o(n).$$

It follows that $K = o_p(n)$, which implies Theorem 3.3.

Acknowledgement

We are grateful for the referee's helpful comments, which have led to this more succinct version of our paper.

References

R.J. Adler, The Geometry of Random Fields (Wiley, New York, 1981).

M.V. Berry, Diffractals, J. Phys. A: Math. Gen. 12 (1979) 781–797.

M.V. Berry and J.H. Hannay, Topography of random surfaces, Nature 273 (1978) 573.

P.H. Carter, R. Cawley and R.D. Mauldin, Mathematics of dimension measurement for graphs of functions, in: D.A. Weitz, L.M. Sander and B.B. Mandelbrot, eds., Fractal Aspects of Materials (Materials Research Society, Pittsburgh, 1988).

A.G. Constantine and P. Hall, Characterising surface smoothness via estimation of effective fractal dimension, J. Roy. Statist. Soc. Ser. B 56 (1994) 97–113.

M. Coster and J.L. Chermant, Recent developments in quantitative fractography, Int. Metals Rev. 28 (1983) 234–238.

L.N. Gilbertson and R.D. Zipp, Fractography and Materials Science (American Society for Testing Materials, Philadelphia, 1981).

P. Hall and R. Roy, On the relationship between fractal dimension and fractal index for stationary stochastic processes, to appear in: Ann. Applied Probab. (1994).

B.B. Mandelbrot, D.E. Passoja and A.J. Paullay, Fractal character of fracture surfaces of metals, Nature 308 (1984) 721–722.

Y. Ogata and K. Katsura, Maximum likelihood estimates of the fractal dimension for spatial patterns, Biometrika 78 (1991) 463–474.

C.C. Taylor and S.J. Taylor, Estimating the dimension of a fractal, J. Roy. Statist. Soc. Ser. B 53 (1991) 353–364.

T.R. Thomas and A.P. Thomas, Fractals and engineering surface roughness, Surface Topography 1 (1988) 143–152.

A.T.A. Wood and G. Chan, Simulation of stationary Gaussian processes in $[0, 1]^d$, Research Report No. CMA-SR06-93, CMA, Austral. Nat. Univ. (Canberra, Austral., 1993).